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An exponential Diophantine equation related to the difference between powers of two consecutive Balancing numbers

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Abstract

In this paper, we find all solutions of the exponential Diophantine equation $B_{n+1}^x - B_n^x = B_m$ in positive integer variables (m, n, x), where B_k is the k-th term of the Balancing sequence.

Keywords: Balancing numbers, Linear form in logarithms, reduction method.

MSC: 11B39, 11J86

1. Introduction

The first definition of balancing numbers is essentially due to Finkelstein [3], although he called them numerical centers. A positive integer n is called a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

holds for some positive integer r. Then r is called the *balancer* corresponding to the balancing number n. For example, 6 and 35 are balancing numbers with balancers 2 and 14, respectively. The n-th term of the sequence of balancing numbers is denoted by B_n. The balancing numbers satisfy the recurrence relation

$$B_n = 6B_{n-1} - B_{n-2}, \text{ for all } n \geq 2,$$

where the initial conditions are $B_0 = 0$ and $B_1 = 1$. Its first terms are

$$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, \ldots$$

It is well-known that

$$B_{n+1}^2 - B_n^2 = B_{2n+2}, \text{ for any } n \geq 0.$$

In particular, this identity tells us that the difference between the square of two consecutive Balancing numbers is still a Balancing number. So, one can ask if this identity can be generalized?

Diophantine equations involving sum or difference of powers of two consecutive members of a given linear recurrent sequence $\{U_n\}_{n\geq 1}$ were also considered in several papers. For example, in [5], Marques and Togbé proved that if $s \geq 1$ an integer such that $F_m^s + F_{m+1}^s$ is a Fibonacci number for all sufficiently large m, then $s \in \{1, 2\}$. In [4], Luca and Oyono proved that there is no integer $s \geq 3$ such that the sum of sth powers of two consecutive Fibonacci numbers is a Fibonacci number. Later, their result has been extended in [8] to the generalized Fibonacci numbers and recently in [7] to the Pell sequence.

Here, we apply the same argument as in [4] to the Balancing sequence and prove the following:

Theorem 1.1. *The only nonnegative integer solutions* (m, n, x) *of the Diophantine equation*

$$B_{n+1}^x - B_n^x = B_m \tag{1.1}$$

are $(m, n, x) = (2n+2, n, 2), (1, 0, x), (0, n, 0)$.

Our proof of Theorem 1.1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [1]. Here, we will use a version due to Dujella and Pethő in [2, Lemma 5(a)].

2. Preliminary results

2.1. The Balancing sequences

Let $(\alpha, \beta) = (3+2\sqrt{2}, 3-2\sqrt{2})$ be the roots of the characteristic equation $x^2 - 6x + 1 = 0$ of the Balancing sequence $(B_n)_{n\geq 0}$. The Binet formula for B_n is

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, \quad \text{for all } n \geq 0. \tag{2.1}$$

This implies that the inequality

$$\alpha^{n-2} \leq B_n \leq \alpha^{n-1} \tag{2.2}$$

holds for all positive integers n. It is easy to prove that

$$\frac{B_n}{B_{n+1}} \leq \frac{5}{29} \tag{2.3}$$

holds, for any $n \geq 2$.

2.2. Linear forms in logarithms

For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{i=1}^{d}\left(X - \gamma^{(i)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log\max\left(1, \left|\gamma^{(i)}\right|\right)\right)$$

the usual absolute logarithmic height of γ.

With this notation, Matveev proved the following theorem (see [6]).

Theorem 2.1. *Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be positive real numbers satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\}, \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1}\cdots\gamma_s^{b_s} - 1 \neq 0$, then

$$|\gamma_1^{b_1}\cdots\gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1\cdots A_s).$$

2.3. Reduction algorithm

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction expansion of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < m\gamma - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \leq M \quad \textit{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. The proof of Theorem 1.1

3.1. An inequality for x versus m and n

The case $nx = 0$ is trivial so we assume that $n \geq 1$ and that $x \geq 1$. Observe that since $B_n < B_{n+1} - B_n < B_{n+1}$, the Diophantine equation (1.1) has no solution when $x = 1$.

When $n = 1$, we get $B_m = 6^x - 1$. In this case, we have that m is odd. Thus, using the Binet formula (2.1), we obtained the following factorization

$$6^x = B_m + 1 = B_m + B_1 = B_{(m+1)/2}C_{(m-1)/2},$$

where $\{C_m\}_{m\geq 1}$ is the Lucas Balancing sequence given by the recurrence $C_m = 6C_{m-1} - C_{m-2}$ with initial conditions $C_0 = 2$, $C_1 = 6$. The Binet formula of the Lucas Balancing sequence is given by $C_n = \alpha^n + \beta^n$. This shows that the largest prime factor of $B_{(m+1)/2}$ is 3 and by Carmichael's Primitive Divisor Theorem we conclude that $(m+1)/2 \leq 12$, so $m \leq 23$. Now, one checks all such m and gets no additional solution with $n = 1$.

So, we can assume that $n \geq 2$ and $x \geq 3$. Therefore, we have

$$B_m = B_{n+1}^x - B_n^x \geq B_3^3 - B_1^3 = 215,$$

which implies that $m > 4$. Here, we use the same argument from [4] to bound x in terms of m and n. Since most of the details are similar, we only sketch the argument.

Using inequality (2.2), we get

$$\alpha^{m-1} > B_m = B_{n+1}^x - B_n^x \geq B_n^x > \alpha^{(n-2)x}$$

and

$$\alpha^{m-2} < B_m = B_{n+1}^x - B_n^x < B_{n+1}^x < \alpha^{nx}.$$

Thus, we have

$$(n-2)x + 1 < m < nx + 2. \tag{3.1}$$

Estimate (3.1) is essential for our purpose.

Now, we rewrite equation (1.1) as

$$\frac{\alpha^m}{4\sqrt{2}} - B_{n+1}^x = -B_n^x + \frac{\beta^m}{4\sqrt{2}}. \tag{3.2}$$

Dividing both sides of equation (3.2) by B_{n+1}^x, taking absolute value and using the inequality (2.3), we obtain

$$\left|\alpha^m (4\sqrt{2})^{-1} B_{n+1}^{-x} - 1\right| < 2\left(\frac{B_n}{B_{n+1}}\right)^x < \frac{2}{5.8^x}. \tag{3.3}$$

Put

$$\Lambda_1 := \alpha^m (4\sqrt{2})^{-1} B_{n+1}^{-x} - 1. \tag{3.4}$$

If $\Lambda_1 = 0$, we get $\alpha^m = 4\sqrt{2} B_{n+1}^x$. Thus $\alpha^{2m} \in \mathbb{Z}$, which is false for all positive integers m, therefore $\Lambda_1 \neq 0$.

At this point, we will use Matveev's theorem to get a lower bound for Λ_1. We set $s := 3$ and we take

$$\gamma_1 := \alpha, \quad \gamma_2 := 4\sqrt{2}, \quad \gamma_3 := B_{n+1}, \quad b_1 := m, \quad b_2 := -1, \quad b_3 := -x.$$

Note that $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$, so we can take $D := 2$. Since $h(\gamma_1) = (\log\alpha)/2$, $h(\gamma_2) = (\log 32)/2$ and $h(\gamma_3) = \log B_{n+1} < n\log\alpha$, we can take $A_1 := \log\alpha$, $A_2 := \log 32$ and $A_3 := 2n\log\alpha$. Finally, inequality (3.1) implies that $m > (n-2)x \geq x$, thus we can take $B := m$. We also have $B := m \leq nx + 2 < (n+2)x$. Hence, Matveev's theorem implies that

$$\begin{aligned}
\log|\Lambda_1| &\geq -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2)(\log\alpha)(\log 32)(2n\log\alpha)(1+\log m)\\
&\geq -2.1 \times 10^{13} n(1+\log m).
\end{aligned} \tag{3.5}$$

The inequalities (3.3), (3.4) and (3.5) give that

$$x < 1.2 \times 10^{13} n(1+\log m) < 2.1 \times 10^{13} n \log m,$$

where we used the fact that $1 + \log m < 1.7\log m$, for all $m \geq 5$. Together with the fact that $m < (n+2)x$, we get that

$$x < 2.1 \times 10^{13} n \log((n+2)x).$$

3.2. Small values of n

Next, we treat the cases when $n \in [2, 37]$. In this case,

$$x < 2.1 \times 10^{13} n \log((n+2)x) < 7.8 \times 10^{14} \log(46x)$$

so $x < 4 \times 10^{16}$.

Now, we take another look at Λ_1 given by expression (3.4). Put

$$\Gamma_1 := m \log \alpha - \log(4\sqrt{2}) - x \log B_{n+1}.$$

Thus, $\Lambda_1 = e^{\Gamma_1} - 1$. One sees that the right-hand side of (3.2) is a number in the interval $[-B_n^x, -B_n^x + 1]$. In particular, Λ_1 is negative, which implies that Γ_1 is negative. Thus,

$$0 < -\Gamma_1 < \frac{2}{5.8^x},$$

so

$$0 < x\left(\frac{\log B_{n+1}}{\log \alpha}\right) - m + \left(\frac{\log(4\sqrt{2}}{\log \alpha}\right) < \frac{2}{5.8^x \log \alpha}. \tag{3.6}$$

For us, inequality (3.6) is

$$0 < x\gamma - m + \mu < AB^{-x},$$

where

$$\gamma := \frac{\log B_{n+1}}{\log \alpha}, \quad \mu = \frac{\log(4\sqrt{2})}{\log \alpha}, \quad A = \frac{2}{\log \alpha}, \quad B = 5.8.$$

We take $M := 4 \times 10^{16}$.

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the condition. In one minute all the computations were done. In all cases, we obtained $x \leq 77$. A computer search with Maple revealed in less than one minute that there are no solutions to the equation (1.1) in the range $n \in [3, 37]$ and $x \in [3, 77]$.

3.3. An upper bound on x in terms of n

From now on, we assume that $n \geq 38$. Recall from the previous section that

$$x < 2.1 \times 10^{13} n \log((n+2)x). \tag{3.7}$$

Next, we give an upper bound on x depending only on n. If

$$x \leq n + 2, \tag{3.8}$$

then we are through. Otherwise, that is if $n + 2 < x$, we then have

$$x < 2.1 \times 10^{13} n \log x^2 = 4.2 \times 10^{13} n \log x,$$

which can be rewritten as

$$\frac{x}{\log x} < 4.2 \times 10^{13} n. \tag{3.9}$$

Using the fact that, for all $A \geq 3$

$$\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A \log A,$$

and the fact that $\log(4.2 \times 10^{13} n) < 10 \log n$ holds for all $n \geq 38$, we get that

$$\begin{aligned} x &< 2(4.2 \times 10^{13} n) \log((4.2 \times 10^{13} n) \\ &< 8.4 \times 10^{13} n (10 \log n) \\ &< 8.4 \times 10^{14} n \log n. \end{aligned} \tag{3.10}$$

From (3.8) and (3.10), we conclude that the inequality

$$x < 8.4 \times 10^{14} n \log n \tag{3.11}$$

holds.

3.4. An absolute upper bound on x

Let us look at the element

$$y := \frac{x}{\alpha^{2n}}.$$

The above inequality (3.11) implies that

$$y < \frac{8.4 \times 10^{14} n \log n}{\alpha^{2n}} < \frac{1}{\alpha^n}, \tag{3.12}$$

where the last inequality holds for any $n \geq 23$. In particular, $y < \alpha^{-38} < 10^{-31}$. We now write

$$B_n^x = \frac{\alpha^{nx}}{32^{x/2}} \left(1 - \frac{1}{\alpha^{2n}}\right)^x$$

and

$$B_{n+1}^x = \frac{\alpha^{(n+1)x}}{32^{x/2}} \left(1 - \frac{1}{\alpha^{2(n+1)}}\right)^x.$$

We have

$$0 < \left(1 - \frac{1}{\alpha^{2n}}\right) < e^y < 1 + 2y,$$

because $y < 10^{-31}$ is very small. The same inequality holds if we replace n by $n+1$. Hence, we have that

$$\max \left\{ \left| B_n^x - \frac{\alpha^{nx}}{32^{x/2}} \right|, \left| B_{n+1}^x - \frac{\alpha^{(n+1)x}}{32^{x/2}} \right| \right\} < \frac{2y\alpha^{(n+1)x}}{32^{x/2}}.$$

We now return to our equation (1.1) and rewrite it as

$$\begin{aligned}\frac{\alpha^m-\beta^m}{4\sqrt{2}} &= B_m = B_{n+1}^x - B_n^x\\ &= \frac{\alpha^{(n+1)x}}{32^{x/2}} - \frac{\alpha^{nx}}{32^{x/2}} + \left(B_{n+1}^x - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right) - \left(B_n^x - \frac{\alpha^{nx}}{32^{x/2}}\right),\end{aligned}$$

or

$$\begin{aligned}\left|\frac{\alpha^m}{32^{1/2}} - \frac{\alpha^{nx}}{32^{x/2}}(\alpha^x-1)\right| &= \left|\frac{\beta^m}{32^{1/2}} + \left(B_{n+1}^x - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right) - \left(B_n^x - \frac{\alpha^{nx}}{32^{x/2}}\right)\right|\\ &< \frac{1}{\alpha^m} + \left|B_{n+1}^x - \frac{\alpha^{(n+1)x}}{32^{x/2}}\right| + \left|B_n^x - \frac{\alpha^{nx}}{32^{x/2}}\right|\\ &< \frac{1}{\alpha^m} + 2y\left(\frac{\alpha^{nx}(1+\alpha^x)}{32^{x/2}}\right).\end{aligned}$$

Thus, multiplying both sides by $\alpha^{-(n+1)x}32^{x/2}$, we obtain that

$$\begin{aligned}\left|\alpha^{m-(n+1)x}32^{(x-1)/2} - (1-\alpha^{-x})\right| &< \frac{32^{x/2}}{\alpha^{m+(n+1)x}} + 2y(1+\alpha^{-x})\\ &< \frac{1}{2\alpha^n} + \frac{396y}{197} < \frac{3}{\alpha^n},\end{aligned} \tag{3.13}$$

where we used the fact that $32^{x/2}/(\alpha^{(n+1)x}) \le (4\sqrt{2}/\alpha^{38})^x < 1/2$, $m \ge (n-2)x \ge n$ and $\alpha^x \ge \alpha^3 > 197$, as well as inequality (3.12). Hence, we conclude that

$$\left|\alpha^{m-(n+1)x}32^{(x-1)/2} - 1\right| < \frac{1}{\alpha^x} + \frac{3}{\alpha^n} \le \frac{4}{\alpha^l}, \tag{3.14}$$

where $l := \min\{n, x\}$. We now set

$$\Lambda_2 := \alpha^{m-(n+1)x}32^{(x-1)/2} - 1 \tag{3.15}$$

and observe that $\Lambda_2 \neq 0$. Indeed, for if $\Lambda_2 = 0$, then $\alpha^{2((n+1)x-m)} = 32^{x-1} \in \mathbb{Z}$ which is possible only when $(n+1)x = m$. But if this were so, then we would get $0 = \Lambda_2 = 32^{(x-1)/2} - 1$, which leads to the conclusion that $x = 1$, which is not possible. Hence, $\Lambda_2 \neq 0$. Next, let us notice that since $x \ge 3$ and $m \ge 38$, we have that

$$|\Lambda_2| \le \frac{1}{\alpha^3} + \frac{1}{\alpha^{38}} < \frac{1}{2}, \tag{3.16}$$

so that $\alpha^{m-(n+1)x}32^{(x-1)/2} \in [1/2, 3/2]$. In particular,

$$(n+1)x - m < \frac{1}{\log\alpha}\left(\frac{(x-1)\log 32}{2} + \log 2\right) < x\left(\frac{\log 32}{2\log\alpha}\right) < x \tag{3.17}$$

and

$$(n+1)x - m > \frac{1}{\log\alpha}\left(\frac{(x-1)\log 32}{2} - \log 2\right) > 0.9x - 1.4 > 0. \tag{3.18}$$

We lower bound the left-hand side of inequality (3.15) using again Matveev's theorem. We take

$$s := 2,\ \gamma_1 := \alpha,\ \gamma_2 := 4\sqrt{2},\ b_1 := m - (n+1)x,\ b_2 := x - 1,$$

$$D := 2,\ A_1 := \log \alpha,\ A_2 := \log 32, \quad \text{and } B := x.$$

We thus get that

$$\log |\Lambda_2| > -1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(\log \alpha)(\log 32)(1 + \log x). \tag{3.19}$$

The inequalities (3.14) and (3.19) give

$$l < 4 \times 10^{10} \log x.$$

Treating separately the case $l = x$ and the case $l = n$, following the argument in [4] we have that the upper bound

$$x < 7 \times 10^{28}$$

always holds.

3.5. Reducing the bound on x

Next, we take

$$\Gamma_2 := (x-1)\log(4\sqrt{2}) - ((n+1)x - m)\log \alpha.$$

Observe that $\Lambda_2 = e^{\Gamma_2} - 1$, where Λ_2 is given by (3.15). Since $|\Lambda_2| < \frac{1}{2}$, we have that $e^{|\Gamma_2|} < 2$. Hence,

$$|\Gamma_2| \le e^{|\Gamma_2|} \left| e^{\Gamma_2} - 1 \right| < 2\,|\Lambda_2| < \frac{2}{\alpha^x} + \frac{6}{\alpha^n}.$$

This leads to

$$\left| \frac{\log(4\sqrt{2})}{\log \alpha} - \frac{(n+1)x - m}{x-1} \right| < \frac{1}{(x-1)\log \alpha}\left(\frac{2}{\alpha^x} + \frac{6}{\alpha^n} \right). \tag{3.20}$$

Assume next that $x > 100$. Then $\alpha^x > \alpha^{100} > 10^{33} > 10^4 x$. Hence, we get that

$$\frac{1}{(x-1)\log \alpha}\left(\frac{2}{\alpha^x} + \frac{6}{\alpha^n} \right) < \frac{8}{x(x-1)10^4 \log \alpha} < \frac{1}{2200(x-1)^2}. \tag{3.21}$$

Estimates (3.20) and (3.21) lead to

$$\left| \frac{\log(4\sqrt{2})}{\log \alpha} - \frac{(n+1)x - m}{x-1} \right| < \frac{1}{2200(x-1)^2}. \tag{3.22}$$

By a criterion of Legendre, inequality (3.22) implies that the rational number $((n+1)x-m)/(x-1)$ is a convergent to $\gamma := \log(4\sqrt{2})/\log\alpha$. Let

$$[a_0, a_1, a_2, a_3, a_4, a_5, a_6, \dots] = [0, 1, 57, 1, 234, 2, 1, \dots]$$

be the continued fraction of γ, and let p_k/q_k be it's kth convergent. Assume that $((n+1)x-m)/(x-1) = p_k/q_k$ for some k. Then, $x-1 = dq_k$ for some positive integer d, which in fact is the greatest common divisor of $(n+1)x-m$ and $x-1$. We have the inequality

$$q_{54} > 7 \times 10^{28} > x-1.$$

Thus, $k \in \{0, \dots, 53\}$. Furthermore, $a_k \le 234$ for all $k = 0, 1, \dots, 53$. From the known properties of the continued fraction, we have that

$$\left|\gamma - \frac{(n+1)x-m}{x-1}\right| = \left|\gamma - \frac{p_k}{q_k}\right| > \frac{1}{(a_k+2)q_k^2} \ge \frac{d^2}{236(x-1)^2} \ge \frac{1}{236(x-1)^2},$$

which contradicts inequality (3.22). Hence, $x \le 100$.

3.6. The final step

To finish, we go back to inequality (3.13) and rewrite it as

$$\left|\alpha^{m-(n+1)x} 32^{(x-1)/2}(1-\alpha^{-x})^{-1} - 1\right| < \frac{3}{\alpha^n(1-\alpha^{-x})} < \frac{4}{\alpha^n}.$$

Recall that $x \in [3, 100]$ and from inequalities (3.17) and (3.18), we have that

$$0.9x - 1.4 < (n+1)x - m < x.$$

Put $t := (n+1)x-m$. We computed all the numbers $\left|\alpha^{-t}32^{(x-1)/2}(1+\alpha^{-x})^{-1} - 1\right|$ for all $x \in [3, 100]$ and all $t \in [\lfloor 0.9x - 1.4 \rfloor, \lfloor x \rfloor]$. None of them ended up being zero and the smallest of these numbers is $> 10^{-1}$. Thus, $1/10 < 3/\alpha^n$, or $\alpha^n < 30$, so $n \le 3$ which is false.

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References

[1] A. Baker, H. Davenport: *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, The Quarterly Journal of Mathematics 20.1 (1969), pp. 129–137, doi: 10.1093/qmath/20.1.129.

[2] A. Dujella, A. Pethő: *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) 49.195 (1998), pp. 291–306, doi: 10.1093/qmathj/49.3.291.

[3] R. P. Finkelstein: *The house problem*, American Math. Monthly 72 (1965), pp. 1082–1088, doi: 10.2307/2315953.

[4] F. Luca, R. Oyono: *An exponential Diophantine equation related to powers of two consecutive Fibonacci numbers*, Proc. Japan Acad. Ser. A 87 (2011), pp. 45–50, doi: 10.3792/pjaa.87.45.

[5] D. Marques, A. Togbé: *On the sum of powers of two consecutive Fibonacci numbers*, Proc. Japan Acad. Ser. A 86 (2010), pp. 174–176, doi: 10.3792/pjaa.86.174.

[6] E. M. Matveev: *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II*, Izv. Math. 64.6 (2000), pp. 1217–1269, doi: 10.1070/im2000v064n06abeh000314.

[7] S. E. Rihane, B. Faye, F. Luca, A. Togbé: *On the exponential Diophantine equation* $P_n^x + P_{n+1}^x = P_m$, To appear in Turkish Journal of Mathematics.

[8] C. A. G. Ruiz, F. Luca: *An exponential Diophantine equation related to the sum of powers of two consecutive* k*-generalized Fibonacci numbers*, Coll. Math. 137.2 (2014), pp. 171–188, doi: 10.4064/cm137-2-3.