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Best bounds for dispersion of ratio block sequences for certain subsets of integers

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Abstract

In this paper, we study the behavior of dispersion of special types of sequences which block sequence is dense.

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1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. Let $X = \{x_1 < x_2 < x_3 < \cdots\}$ be an infinite subset of $\mathbb{N}$. Denote by $R(X) = \{\frac{x_i}{x_j} : i, j \in \mathbb{N}\}$ the *ratio set of* X, and say that a set X is (R)-*dense* if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. The concept of (R)-density was introduced by T. Šalát [7].

The following sequence of finite sequences derived from X

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1.1}$$

is called *the block sequence* of the sequence X.

It is formed by the blocks $X_1, X_2, \dots, X_n, \dots$ where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

is called the *n-th block*. This kind of block sequences was introduced by O. Strauch and J. T. Tóth [9].

For each $n \in \mathbb{N}$ consider the *step distribution function*

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

and define the *set of distribution functions of the ratio block sequence*

$$G(X_n) = \{\lim_{k\to\infty} F(X_{n_k}, x)\}.$$

The set of distribution functions of ratio block sequences was studied in [1, 2, 5, 6, 8, 12].

For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\},$$

the maximum distance between two consecutive terms in the n-th block. We will consider the quantity

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n),$$

(see [10]) called the *dispersion* of the block sequence (1.1) derived from X. Relations between asymptotic density and dispersion were studied in [11].

The aim of this paper is to study the behavior of dispersion of the block sequence derived from X under the assumption that $X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ is (R)-dense and the limit $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = s$ exists. In this case

$$\underline{D}(X) \le \begin{cases} \frac{1}{s+1}, & \text{if } s \in \left\langle 1, \frac{1+\sqrt{5}}{2}\right), \\ \frac{1}{s^2}, & \text{if } s \in \left\langle \frac{1+\sqrt{5}}{2}, 2\right), \\ \frac{s-1}{s^2}, & \text{if } s \in \langle 2, \infty) \end{cases}$$

(see [10, Theorem 10]). This upper bound for $\underline{D}(X)$ is the best possible if $s \ge 2$ (see [4]) and in the case $\frac{1+\sqrt{5}}{2} \le s \le 2$ (see [3]). We prove that the above upper bound for $\underline{D}(X)$ is also optimal in the remainding case $1 \le s < \frac{1+\sqrt{5}}{2}$, i.e. $\underline{D}(X)$ can be any number in the interval $\langle 0, \frac{1}{s+1}\rangle$.

2. Results

First, we show that there is a connection between the dispersion and the distribution functions of a ratio block sequence.

Theorem 2.1. *Let $X \subset \mathbb{N}$, and assume that the dispersion $\underline{D}(X)$ of the related block sequence is positive. Let $g \in G(X_n)$. Then g is constant on an interval of length $\underline{D}(X)$.*

Proof. Let $\varepsilon < \underline{D}(X)$ be an arbitrary positive real number. By the definition of dispersion it follows that for sufficiently large n the step distribution function $F(X_n, x)$ is constant on some interval $\left(\frac{x_i}{x_n}, \frac{x_{i+1}}{x_n}\right\rangle$ of length $\underline{D}(X) - \varepsilon$. A simple compactness argument yields that there exist

- real numbers $\gamma,\ \delta \in \langle 0, 1\rangle$ such that $\delta - \gamma \geq \underline{D}(X) - \varepsilon$,
- an increasing sequence (n_k) and a sequence (m_k) of positive integers such that $m_k < n_k$,

$$\lim_{k\to\infty} \frac{x_{m_k}}{x_{n_k}} = \gamma, \quad \lim_{k\to\infty} \frac{x_{m_k+1}}{x_{n_k}} = \delta \quad \text{and} \quad \lim_{k\to\infty} F(X_{n_k}, x) = g(x) \text{ a.e. on } \langle 0, 1\rangle.$$

Hence g is constant on the interval (γ, δ) of length $\underline{D}(X) - \varepsilon$. Since ε can be chosen arbitrary small, and the assertion of the theorem follows. □

The next lemma is useful for the determination of the value of the dispersion $\underline{D}(X)$ (see [10, Theorem 1]).

Lemma 2.2. *Let*

$$X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N},$$

and for $n \in \mathbb{N}$ let $c_n < d_n < c_{n+1}$ be positive integers. Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1, \dots, n\}}{d_{n+1}}.$$

For the proof of (R)-density we shall use the following lemma.

Lemma 2.3. *Denote by (p_n), (q_n), (u_n), (v_n), (w_n) and (z_n) be strictly increasing sequences of positive integers satisfying*

$$p_n < q_n < u_n < v_n \quad \text{and} \quad w_n < z_n, \quad (n = 1, 2, 3, \dots).$$

Further, let

$$\left(\frac{q_n}{p_n}\right), \ \left(\frac{u_n}{p_n}\right), \ \left(\frac{v_n}{u_n}\right), \ \left(\frac{w_n}{u_n}\right) \text{ and } \left(\frac{z_n}{w_n}\right)$$

converge to real numbers greater than 1, moreover

$$\lim_{n\to\infty} \frac{z_n}{w_n} \geq \lim_{n\to\infty} \frac{u_n}{q_n}.$$

Then the ratio set of

$$\bigcup_n \big((p_n, q_n\rangle \cup (u_n, v_n\rangle \cup (w_n, z_n\rangle \big) \cap \mathbb{N}$$

is dense on the interval

$$\left\langle \lim_{n\to\infty} \frac{w_n}{v_n}, \lim_{n\to\infty} \frac{z_n}{p_n} \right\rangle.$$

The proof is elementary and we leave it to the reader. Let us suppose that $k \in \mathbb{N}$ is a constant. Note that the assertion of the lemma remains still true if one removes k elements from the sets $(u_n, v_n\rangle \cap \mathbb{N}$ for all sufficiently large n.

The main result of this paper is the following.

Theorem 2.4. *Let $s \in (1, \frac{1+\sqrt{5}}{2})$ be an arbitrary real number. Then for any $\alpha \in \langle 0, \frac{1}{s+1}\rangle$ there is an (R)-dense set*

$$X = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N},$$

where $c_n < d_n < c_{n+1}$ are positive integers for that $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = s$ and $\underline{D}(X) = \alpha$.

Proof. It was shown in [4, Theorem 2] that the dispersion $\underline{D}(X)$ can take any number in the interval $\langle 0, \frac{s-1}{s^2}\rangle$. In what follows we suppose $\frac{s-1}{s^2} < \alpha \le \frac{1}{s+1}$. Let us consider the function $f(x) = \frac{x-1}{sx}$. Clearly, this function is continuous and increasing on the interval $\langle 1, \infty)$. Moreover

$$f(s) = \frac{s-1}{s^2} \quad \text{and} \quad f(s+1) = \frac{1}{s+1}.$$

Thus, there exists a real number $t \in (s, s+1\rangle$ with the property

$$\frac{t-1}{st} = \alpha. \tag{2.1}$$

Write $\frac{1}{\alpha}$ in the form $s^{k+\delta}$, where k is an integer and $0 \le \delta < 1$. The lower bound $k \ge 2$ follows from the facts that $s+1 \le \frac{1}{\alpha}$ and $s+1 \ge s^2$ whenever $1 < s \le \frac{1+\sqrt{5}}{2}$.

Define the set $X \subset \mathbb{N}$ by

$$X = \bigcup_{n=1}^{\infty} \big(A_n \cup B_n\big) \cap \mathbb{N},$$

where

$$A_n = \bigcup_{i=1}^{k} (a_{n,i}, b_{n,i}\rangle \quad \text{and} \quad B_n = \bigcup_{j=1}^{n} (c_{n,j}, d_{n,j}\rangle.$$

Put $a_{1,1} = 1$ and

$$b_{n,i} = [s.a_{n,i}] \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad i = 1, 2, \dots, k,$$

$$a_{n,i} = \begin{cases} d_{n-1,n-1}! & \text{for } n \ge 2,\ i = 1 \\ [s^{\delta}.b_{n,1}] + 1 & \text{for } n \in \mathbb{N},\ i = 2 \\ b_{n,i-1} + 1 & \text{for } n \in \mathbb{N},\ i = 3, \dots, k, \end{cases}$$

$$c_{n,j} = \begin{cases} [t.b_{n,k}] + 1 & \text{for } n \in \mathbb{N},\ j = 1 \\ [t.d_{n,j-1}] + 1 & \text{for } n \in \mathbb{N},\ j = 2, \dots, n, \end{cases}$$

$$d_{n,j} = [s.c_{n,j}] \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad j = 1, 2, \ldots, n.$$

First we prove that $\underline{D}(X) = \alpha$. For sufficiently large n, by the definition of the set X we have the inequalities

$$\begin{aligned} a_{n+1,1} - d_{n,n} > c_{n,n} - d_{n,n-1} > c_{n,n-1} - d_{n,n-2} > \cdots \\ > c_{n,3} - d_{n,2} > c_{n,2} - d_{n,1} > c_{n,1} - b_{n,k}, \end{aligned} \tag{2.2}$$

further

$$a_{n,1} - d_{n-1,n-1} < c_{n,1} - b_{n,k} \tag{2.3}$$

and

$$a_{n,2} - b_{n,1} < a_{n,1} - d_{n-1,n-1}. \tag{2.4}$$

Observe that inequality (2.3) holds if $\frac{1}{\alpha}(t-1) > 1$. In virtue of (2.1) this is equivalent with $st > 1$, which evidently holds. As

$$s^{1+\delta} - s - 1 < s^2 - s - 1$$

and $s^2 - s - 1$ is negative for $s \in (1, \frac{1+\sqrt{5}}{2})$, inequality (2.4) follows.

Now we use Lemma 1. From the inequalities (2.2–2.4) one can see that it is sufficient to study the quotients

$$\text{a)}\ \frac{a_{n,1} - d_{n-1,n-1}}{b_{n,k}}, \quad \text{b)}\ \frac{c_{n,1} - b_{n,k}}{d_{n,1}}, \quad \text{c)}\ \frac{c_{n,k} - d_{n,k-1}}{d_{n,k}}.$$

In case a) we see

$$\liminf_{n\to\infty} \frac{a_{n,1} - d_{n-1,n-1}}{b_{n,k}} = \liminf_{n\to\infty} \frac{a_{n,1}}{\frac{1}{\alpha} a_{n,1}} = \alpha,$$

in case b)

$$\liminf_{n\to\infty} \frac{c_{n,1} - b_{n,k}}{d_{n,1}} = \liminf_{n\to\infty} \frac{tb_{n,k} - b_{n,k}}{stb_{n,k}} = \frac{t-1}{st} = \alpha,$$

and the remaining case c) is analogous to case b).

It remains to prove that the set X is (R)-dence. Using Lemma 2 we show that the ratio set of the set X is dense on the intervals

$\left\langle 1, \dfrac{1}{\alpha} \right\rangle$ (for $p_n = a_{n,1}, q_n = b_{n,1}, u_n = w_n = a_{n,2}, v_n = z_n = b_{n,k}$),

$\left\langle t^i s^{i-1}, \dfrac{1}{\alpha} s^i t^i \right\rangle$ (for $p_n = a_{n,1}, q_n = b_{n,1}, u_n = a_{n,2}, v_n = b_{n,k}, w_n = c_{n,i}, z_n = d_{n,i}$).

Hence, by $\frac{1}{\alpha} \geq s + 1$ and $t < s + 1$ we have

$$\left\langle 1, \frac{1}{\alpha} \right\rangle \cup \bigcup_{i=1}^{\infty} \left\langle t^i s^{i-1}, \frac{1}{\alpha} s^i t^i \right\rangle = \langle 1, \infty),$$

and therefore the (R)-density of the set X follows. □

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