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Equivalence relation as a tool to create new structures How could they be prepared and taught in schools?

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In this paper, we look for responses to ideas of introducing the teaching of number and vectors using equivalence relations with children 10-13 years old. We have to express that our goal with this paper is to wake up the interest of teachers and researchers only and we have not elaborated a concrete teaching material or textbook part for this level. We analysed the problems with vectors and rational numbers from the *theoretical point of view* including correct mathematical thoughts. Therefore our work could be seen as *a starting point* for developing new materials and experiments with pupils 10-13 years old. Our interest for this topic was awakened by personal experiences of the authors from the teacher training at university level. We experienced, that students are very confused about vectors and directed segments even as rational numbers and fractions. The relation between these notions is not clear.

In the textbooks used in Hungary and Slovakia, equivalence relations are not *explicitly* mentioned when introducing pupils to such ideas as rational numbers (fractions) and vectors, where the equivalence relation plays a crucial role. The reason for this, in our opinion, is that the authors of those textbooks think (otherwise well) that the notion of equivalence relation is too complicated for pupils at that stage. It is right but followed the ideas T. Varga the complex mathematical notions should be introduced as early as possible to pupils. However, given that there are problems with the introduction of the notions of rational numbers and vectors in our countries, in this article we try to argue against this trend, showing how we could teach those ideas to students and using equivalence relations from an early age and going step by step, developing the idea leading to correct notions later on.

In the first part of the article we show, using textbooks from Hungary and Slovakia, how the notions of rational numbers and vectors are currently introduced in school mathematics. In the second part, we discuss supporting ideas from psychology and cognitive science related to the building of a concept. The third part consists of exercises for simple structures we can rely on in giving tasks for practice, by the help of which the conceptual progress related to the rational numbers might be promoted. In the fourth part, we summarize the ideas of a framework we have used when discussing these ideas with prospective teachers of mathematics.

1. Textbooks

In Hungary, there are many different textbook families from which, normally, the teacher chooses after a common decision within the department of mathematics. In Slovakia, there is a centralized educational system for textbooks, where, usually, only one textbook exists for all students of the same grade. Therefore, our examples are more varied in Hungary.

1.1. Hungarian examples

Firstly,

> A rational number is such a *number* which can be written in the form $\frac{a}{b}$ where both a and b are whole numbers (and $b \neq 0$). See e.g. in Hajdu, S. [7, p. 18], Csatár, K. [5, p. 28] [translation Vancsó, Ö.]

It is not clear what the connection is between the terms fraction and rational number. Another weakness of this definition is that it assumes we know what *number* means. But this is the main problem. We want to extend the notion of number (from whole to rational) and in such a case we can only use the earlier notion of number and not the new one. This is a typical mistake repeated later in the case of real numbers. An exception is where the "value" of a fraction is introduced as a rational number (Vancsó, Ö. [10, p. 69]). This is correct but it uses a non-mathematical notion, which needs to be mathematized later.

Secondly,

> A vector is a segment with a given direction. See Csatár, K. [6, p. 199.], Koller, M. [8, p. 224], Korányi, E. [9, pp. 279–281].

In this sentence, two different notions are identified, and, with the exception of a single book (see [11, p. 253]), there is a similar introduction in all textbooks. It is interesting that both [8] and [9] introduce the idea of a vector connecting it to the idea of translation but mixing the notions of directed segment and vector. The definition says a vector is an equivalence class of segments that have the same direction and the same length. There is not elaborated enough clearly the difference between the translation as a "global" transformation of a plane (space) and the result of this translation for each one points.

1.2. Slovakian examples

The main focus in the current mathematical books in the field of teaching rational numbers is dual. On the one hand the emphasis is on the display when two different notations (their forms) mean the same rational number. On the other hand the emphasis is on the possible ways of comparison by size of rational numbers having different forms and not the least to make the students able to determine the image of rational numbers on the number line.

The tasks and examples of the given chapter also focus on this. They mention the reduced form of rational numbers, but the fact that the reduced form can be seen as the class representative of the given rational number is not showed. The formulation that one point of the number line is the image of infinitely many but equivalent fractions is also included. The various fractions express the same rational number. The fact, that a rational number can be expressed with the help of different fractions, is also included, but the proof of that is not included according to the teachers. It is said, that the result of the operations does not depend on which form of the fractions are used, by which the rational number is expressed.

Firstly, in summary, we can say that the set of rational numbers is introduced in a way that makes it unclear what kinds of numbers constitute the set (see Sedivy a. coll. [12, pp. 3335], and Sedivy a. coll. [13, pp. 28-35]).

Fractions and rational numbers are dealt with in the 6th-7th grades (11-12 year olds) without mentioning that rational numbers come from classifying fractions. This might be the kernel of the problem. Even at university level, students are still not quite aware of what fractions or rational numbers are. On the other hand, the textbook speaks about the value of the fraction, which is vividly described and illustrated, though mathematically not a well-established notion. Basically the value of a fraction should be made mathematical, which could then lead to the notion of rational numbers. These two notions follow each other, thus certain topic headings speak about rational numbers, while the subheadings are about operations with fractions. Serious misunderstandings may be created because the relationship between these notions is not clarified. The notion of rational numbers is not explained in any form, thus, in place of the rules referring to the operations with rational numbers, there are always fractional operations.

Secondly,

"It is a rarity that a notion is surrounded by so many mysteries, and

> carry so many meanings as the notion of the vector. The substance of the hardships might come from the fact that we can manipulate with single numbers but the world surrounding us cannot be described by single numbers. If a single number is not sufficient, then we try to describe the situation by more numbers and we can speak about vectors right away... For us there will always be only one meaning for the vector and this is a shift." (Discussion of vectors in the 3rd class (16-17 year olds) of secondary schools Hecht T. [15, pp. 25] (translation Lehocka, Z.)

It is fairly confusing not to explain why numbers make a vector; on the other hand the notion is identified clearly with a geometrical transformation which might be hard to understand at first hearing (see Hecht T. [14, pp. 24–25]). One positive aspect of this introduction is that later on this notion can be developed and its relation to directed segments becomes clear. This differs from the Hungarian textbooks, where a vector is identified falsely with a transformation which is undoubtedly not identical to a directed segment. Of course it might be represented that way (see later).

2. Building concepts

The understanding of the equality of fractions, i.e. the fact that different fractions can express the same quantity, is crucial in the linking of quantities and fractions, and also in the perspective of adding and subtracting fractions. Researches show that fraction equivalence is not easy to understand for all students.

The wish to become acquainted with the surrounding world develops in children at an early age. As teachers, we have to build on children's natural curiosity and interest. An interesting task or a problem situation tailored to the child's level of development may challenge them and mobilize their inventiveness. Tamás Varga [4] said the following: "if the child comes to know the geometrical phenomena through aesthetic experience, if we let them learn mathematics through their toys, this world will not be a strange land to them. The connections between thinking and observations, activity-based experience, clearly indicate that children learn how to see, observe, solve tasks and think via actions, that is, independent activities. Teachers who are open, accepting and affirmative towards children's exhibitions of initiative and inventiveness contribute to the development of thinking. We can strengthen in children their feeling of self-trust, initiative and a need for searching out, new ways as well as finding suitable solutions among various possibilities."

According to Bruner, teaching must be built on the structures of mathematics. The advantage of this method is a more easily comprehensible syllabus if students understand the basics; single things and details are soon forgotten if they are not treated in structured forms; understanding basic phenomena contributes to transferring effect; and the difficulties of transmission to higher grades of schools are lessened. Processing the syllabus in an intuitive form in an early phase as well as a later re-discussion of the given topic on a level that suits the development of a certain age group is necessary for a more complex understanding of that syllabus by students.

Concepts are created by putting certain objects in one group according to their existing common characteristics (abstraction). For example, the family's car and the neighbour's car are objects which are alike (since they have a car-body, four seats, can be driven, and can get us to faraway places). A common word „car" has been made for the naming (classifying) of these objects. Naming (classifying) is as important as the concept creating process itself. The primary concepts abstracted from the objects might also be classified according to their existing common characteristics creating secondary concepts (cars, buses and bicycles are all vehicles). Third-level concepts may be abstracted from the second-level concepts and from those even higher level concepts. During the abstraction process it is more expedient to illustrate the concept by examples instead of definitions. Text in facts is the only viable way.

For example, when we say to a child that the colour red is a sensation which is created by the rays of light of about 0.6 micron wavelength reaching the eyes, they cannot create the concept red and cannot say whether a given car is red or not. On the other hand, if we show a lot of examples of red, after some time they will create the concept of red themselves.

In the article we undertake a theoretical concept to the introduction of the rational number and vector terms with the help of equivalence classes. The basic idea of our concept is that, by gradual building, the design of these terms is done intuitively. Meanwhile it is lucky if the students understand the concepts in context.

Our goal is not to create new definitions or theses. We rather aim to outline a possible way in which, by suitable tasks, certain mathematical concepts can be introduced and substantiated.

According to Richard Skemp [3, p. 20-45], in mathematics, the only way of teaching concepts which students have not acquired yet is to help children organize a suitable set of examples into one group in their mind.

If the new concept is a kind of primary concept such as for example, red, we can do this without using any symbols, simply by pointing. If the concept is a secondary one such as all mathematical concepts, then the only method of helping the student collect the set of suitable examples into one group in their mind is collecting the suitable words.

Definitions are not completely useless because they close scientific arguments, ensuring the possibility of unambiguous classification as well as ensuring the proper positioning of the concept in the structures of the concepts of the individuals. However their function is only secondary in the original creation of the concepts.

Sciences work with many concepts and the basic concepts have to be learnt at primary and secondary schools. Without understanding the first concepts, anything built on it, will not be comprehended. If we do not understand which phenomenon/situation is covered in the first concept we will not understand the secondary- and third-level concepts built on the first one. For example, if the concept of the circle is not clear (or if a student does not understand what a set of points equally distant from a given point means), then the teacher will not be able to teach children the triangle's circumcircle either.

Creating clear concepts, seeing the essence and setting up unambiguous and simple models are important functions of a teacher. Clear concepts are needed for logically proper statements. Existence of clear concepts means that students find adequate concepts for various situations. It is also important that students can differentiate two close concepts, for instance parallelogram and trapezium, and as will be discussed later directed segment and vector.

3. Constructing structures by using equivalence relations

Mathematics textbooks were examined from the viewpoint of dealing with the relation between fractions and rational numbers as well as vectors and directed segments. The idea of equivalence behind them and the building of structures are usually not stated explicitly, thus missing the clearing up of the notions and their relations. How could we avoid these mistakes? This is an important goal of this article. Let us give an example to clarify our situation.

The first step can be made in primary schools where all notions are introduced by classification. The essence is always to follow this classification, which means the pupils have to understand how well defined "classes"could be constructed from difference things, objects. We have to know which things belong to a certain group and we can categorize all things in such a way.

3.1. The introduction of equivalence relations and classes

Equivalence classes should be introduced in early ages, grades 1-3 (6-8 years olds).

Example: take a set of words (more or fewer words, depending on the age), for example {Johnny, more cunning, horse, wonderful, pen, intimate, Adam}. Students are asked to group the words according to the numbers of vowels. So, the words with equal numbers of vowels will belong to the same group. As a support, we should use a set of letters or a magnetic ABC set so that students can find the solution by manipulation. The set is freely expandable and we can play with the idea that this is a real classification (i.e. as relations, they are reflexive, symmetrical and transitive, since this is the condition of becoming a real classification).

Let H be a set of a plane's lines. Let us examine the features of the parallel relation on this set H. We can use a match-stick, a drinking-straw and a bigger cardboard sheet. Firstly students examine mutual relations of one, two then three lines. They will experience intuitively that this relation is reflexive, symmetrical and transitive, thus a classification on the set of the plane's lines. The classification might be identified with the direction; therefore there are as many classes as many different directions. This example is more complex, two steps further in abstraction than the earlier one was. The number of classes is infinite, and the pupils can draw them but they cannot manipulated them.

To illustrate this structure we give three different examples, which help to understand how new structures among equivalence classes can be built up using the old one among the original elements of the classes.

3.2. Finite constructions

3.2.1. Residue classes by the division with a given positive whole number

In a more simple case (compared to rational numbers or vectors) we consider a finite number of classes.

It is useful to make pupils do many exercises with this case.

Example 3.1. Divisibility by 2. There are only two classes: the even and the odd numbers (for example: 4 or 13). We can define the "sum" of two classes since the sum of two even and two odd numbers is always even and the sum of one odd and one even number is odd. So, regardless of the representative elements of the classes always the same result occurs. If we note these classes for example by E(ven) and O(dd) then our addition rules can be expressed by: $\mathrm{E}+\mathrm{E}=\mathrm{E}$, $\mathrm{O}+\mathrm{E}=\mathrm{O}$ and $\mathrm{O}+\mathrm{O}=\mathrm{E}$. Now we have to analyse the traditional rules of addition: associativity, commutativity, and the existence of neutral element and inverse. The first two are proved very easily, the neutral element is E, because: $\mathrm{E}+\mathrm{E}=\mathrm{E}$, $\mathrm{O}+\mathrm{E}=\mathrm{O}$. The inverse element is the same as the element itself: $\mathrm{E}+\mathrm{E}=\mathrm{E}$, consequently $\mathrm{E}^{-1}=\mathrm{E}$ similar way $\mathrm{O}+\mathrm{O}=\mathrm{E}$ so $\mathrm{O}^{-1}=\mathrm{O}$. (We call such a structure a group.)

The other operation is multiplication. We remark that if two odd numbers are multiplied, always an odd number comes out. In the other three cases always even numbers come out. It means the following rules: $\mathrm{O}\cdot\mathrm{O}=\mathrm{O}$, $\mathrm{E}\cdot\mathrm{O}=\mathrm{O}\cdot\mathrm{E}=\mathrm{E}\cdot\mathrm{E}=\mathrm{E}$.

It's useful to give children tasks which highlight the rules of associativity and commutativity and help them recognize these rules. There is a unit element (O), because $\mathrm{O}\cdot\mathrm{O}=\mathrm{O}$ and $\mathrm{E}\cdot\mathrm{O}=\mathrm{E}$. It is interesting that among the whole numbers, which are elements of these two classes, there is not an inverse element but in our new structure there is. It means that this is a field.

Such examples and tasks can be posed for primary school children as well.

Example 3.2. Division by 5.

There are five classes here, which means much more complicated operation tables (see below). These tables can be constructed by pupils of grades 6-7 (12-13 years olds). They can check that the class operations are independent of choosing the representative elements from the classes. The two tables are Table 1 and Table 2.

In the general case the division goes by n (this case is $n=5$) but this is too abstract for primary-school pupils, it is for secondary school students of grades 11 or 12 (17-18 years olds).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 1: Addition

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Table 2: Multiplication

These ideas are intended for further elaboration for teachers and textbook writers and for structures by which students might be more activate, and by which we could prepare the operations among the desired equivalence classes, hereby approximating our aims.

3.2.2. Iterated sums of the digits of natural numbers

This is an interesting example which appears to have an even more complicated structure than examples 3.1 and 3.2.

We will represent the natural numbers by $1, 2, 3, \ldots, 9$ the following way. In the first step we sum all the digits of the given number. If the result is less than 10, the process is finished. If not, it will be repeated. After the final step we reach a number between 1 and 9. This construction classifies all positive integers into 9 different classes, and induces an equivalence relation on them as it shown below.

An example: 198564218 is equivalent to 3752, since $1+9+8+5+6+4+2+1+8 = 44$ and after the second step $4+4$ we get 8, similarly $3+7+5+2 = 17$, and after the second step $1 + 7 = 8$. It is easy to see that this is indeed an equivalence relation (simply because it is based on a partition, which always induces an equivalence relation). Furthermore, we may construct a "number type" structure on the set of these nine classes according to the original sum and product operations given on the set of natural numbers. We offer this example for secondary students of grades 10-11 (16-17 years of age).

The operations introduced according to this equivalence relation are well-defined, because the classes are actually the remainder classes of division by 9. The reason of this is the fact that a positive integer has same remainder after division by 9 as the sum of its digits. Therefore the result of an operation does not depend on choosing the representative elements from the classes. For example, the sum of classes "1" and "2" is the class "3". The question is whether sum always belongs to class "3", if we choose other representative numbers from both classes "1" and "2". Example: $46 \to 10 \to 1$; $89471 \to 29 \to 11 \to 2$; $46 + 89471 = 89517 \to 30 \to 3$, it demonstrates that (at least in this case) the operation works well. The case of multiplication is similar. We believe this example is suitable for creative students and provides a possibility to exercise building mathematical structures, in this case by using number theoretic methods.

3.3. Rational numbers

Young children are already able to organise things in different classes. For example, vehicles or games. These experiences can be extended in the case of fractions. A fraction is a relation between two natural numbers (a ratio). It is possible to create classes among fractions. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ belong to one class if and only if the following two products are the same: $ad = cb$. (That means the "value" of the two fractions are the same but this is not a mathematical notion.) An important step is to show that this relation is really a classification, meaning mathematically to fulfill three assumptions: reflexivity, symmetry and transitivity. The first is very formal for pupils it is enough to deal with symmetry and transitivity. Symmetry can be proven very easily (because multiplication is symmetrical), the second is a bit more complex. If $\frac{a}{b}$ and $\frac{c}{d}$ and also $\frac{c}{d}$ and $\frac{e}{f}$ belong to the same class, then it is true that $\frac{a}{b}$ and $\frac{e}{f}$ also belong to the same class. To prove this we have to write the assumptions in the form of equations: $ad = cb$ and $cf = ed$. We have to derive from them that $af = eb$. In order to derive this let the two given equations be multiplied: $(ad)(cf) = (cb)(ed)$. Using two rules of multiplication, associativity and commutativity we then get $afcd = ebcd$. Dividing through by cd we have finished the proof.

Of course, in school such tasks have to be posed where the connection with concrete numbers can be seen. There are links again to number theory and divisibility.

The next and most complicated step is the introduction of addition and multiplication between classes. Our goal is *to regard one class* as *a "new number"*. To do this we have to formulate the operations. We wanted to use the operations of fractions. This is the correct way if we can prove that these operations are well defined because the result is independent of which element of a class was chosen.

a) Addition

Let A and B be two classes of fractions. We define C as a "sum" of these classes in such a way that if $\frac{a_1}{a_2} \in A$ and $\frac{b_1}{b_2} \in B$, then C will be the class where the fractions $\frac{a_1 b_2 + b_1 a_2}{a_2 b_2}$ belong. It is a good definition only if the result does not depend on the representatives of A and B. To prove it we have to show, if $\frac{a_1^*}{a_2^*} \in A$ and $\frac{b_1^*}{b_2^*} \in B$, then $\frac{a_1^* b_2^* + b_1^* a_2^*}{a_2^* b_2^*} \in C$ is true.

In order to prove it we have to point out that

$$(a_1^* b_2^* + b_1^* a_2^*)(a_2 b_2) = (a_1 b_2 + b_1 a_2)(a_2^* b_2^*)$$

supposing that $a_1a_2^* = a_1^*a_2$ and $b_1b_2^* = b_1^*b_2$. The following equation chain shows why $\frac{a_1^*b_2^*+b_1^*a_2^*}{a_2^*b_2^*} \in C$ is true. $(a_1^*b_2^* + b_1^*a_2^*)(a_2b_2) = a_1^*b_2^*a_2b_2 + b_1^*a_2^*a_2b_2 = a_1a_2^*b_2^*b_2 + b_1b_2^*a_2^*a_2 = (a_1b_2 + b_1a_2)(a_2^*b_2^*)$. Therefore the addition is well defined.

b) Multiplication

Let A and B are two classes of fractions. We define C as a "product" of these classes in the following way: if $\frac{a_1}{a_2} \in A$ and $\frac{b_1}{b_2} \in B$ then C will be the class where the fractions $\frac{a_1b_2}{a_2b_2}$ belong. Our goal is again to show, that this is a good definition meaning if $\frac{a_1^*}{a_2^*} \in A$ and $\frac{b_1^*}{b_2^*} \in B$ then $\frac{a_1^*b_1^*}{a_2^*b_2^*} \in C$ as well. We know that $a_1a_2^* = a_1^*a_2$ and $b_1b_2^* = b_1^*b_2$. We have to prove that $a_1b_1a_2^*b_2^* = a_1^*b_1^*a_2b_2$. On the left-hand side, we can replace $a_1a_2^*$ by $a_1^*a_2$, then $a_1b_1a_2^*b_2^* = a_1^*a_2b_1b_2^* = a_1^*b_1^*a_2b_2$ since $b_1b_2^* = b_1^*b_2$. This means that the multiplication is independent of the representative elements, consequently it is correctly defined.

In school we do not think these abstract proofs should be derived, only shown by concrete number examples. We have just proved the operations are well defined. The next step is, if we would like to regard these classes as numbers, to check the usual rules of these operations. This means that for multiplication it is a commutative group similarly to addition. Furthermore, multiplication is the distributive with respect to addition. Most of these rules can be derived by using similar rules of whole numbers. The only exception is the inverse element for multiplication (this does not exist for whole numbers). We can prove easily that $A^{-1} = B$ where B is defined by the following way: if $\frac{a}{b} \in A$ then $\frac{b}{a} \in B$. $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$, since $\frac{a}{b} \cdot \frac{b}{a} = 1$ and the class which contains 1 is the identity element for multiplication. Of course the identity element also has an inverse which is the identity element itself, because only the fractions written in form $\frac{a}{a}$ belong to class "1".

In this way, the student has to understand the difference between fractions and rational numbers and see an important example of how a new structure can be constructed by using an older structure.

3.4. Vectors

There are at least two different ways to introduce vectors. One way is geometrically, using the translation as a transformation (as in the Slovakian textbook) and it is briefly written at the end of chapter 1.1. The first step is on this way is to distinguish between the transformation as a function whose domain is the points of a plane (or space) and the range is the same plane (space) and the concrete operation from points to points. Each point and its image define a directed segment. These are parallel, have the same lengths and direction. One of them can represent the translation as a transformation. This distinction is the most important to avoid the misconceptions.

The notion of vector, is perhaps more confusing than the notion of rational numbers. Among fractions there are operations which are the basis of the operation of their classes. Among directed sections which are the elements of the classes

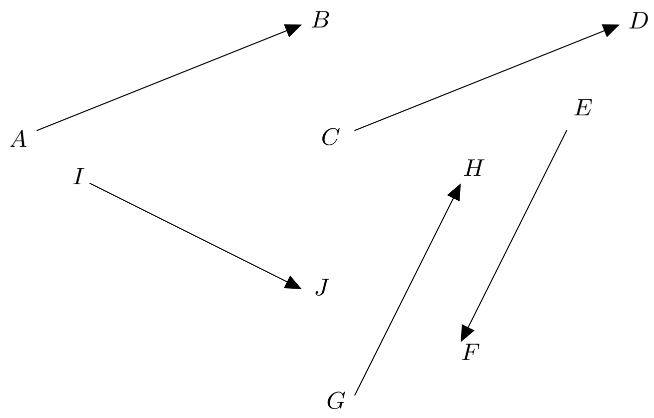

Figure 1

(named later vector) there is only addition in restricted situations, the other operation, named scalar multiplication, is defined without any restriction.

We would like to follow another way, using directed sections and equivalence classes as before.

Our starting point is the set of all directed sections and two operations between them: addition and scalar multiplication (which means a directed segment can be multiplied by a real (or earlier by a rational) number. Our way is: first define a vector as a class of directed segment. Two directed segments are equal (equivalent) if they are parallel, have the same direction and the same lengths (to put it briefly, using a translation they can be rapped into each other). The next step in our case is to define addition and multiplication between classes and to show they are well-defined operations. First step: introduce an equivalence relation $\approx$ the above mentioned way.

In Figure 1 $\overrightarrow{AB} \approx \overrightarrow{CD}$, but only these two sections are equivalent.

It is easy to prove that this relation is really equivalence. We then introduce addition among classes. We have to choose such representatives of the classes which either have common starting points or the ending point of one is the same as the second's starting point, see Figure 2.

To prove this addition is well-defined we have to use only translations, if we have chosen other representatives, the whole picture will be the "same" only translated (see figure 3). This shows that addition is a well-defined operation on the equivalence classes of directed sections. One class can be regarded as a vector. The above operation has important characteristics which are those of a commutative group. After this operation another can be defined as well, the so called multiplication by a scalar (see figure 4). Here it is again easy to prove the well-defined operation. The translation plays the crucial role again.

An easily provable rule of this operation is: $\lambda(\mu\vec{a}) = (\lambda\mu)\vec{a}$. The connection of

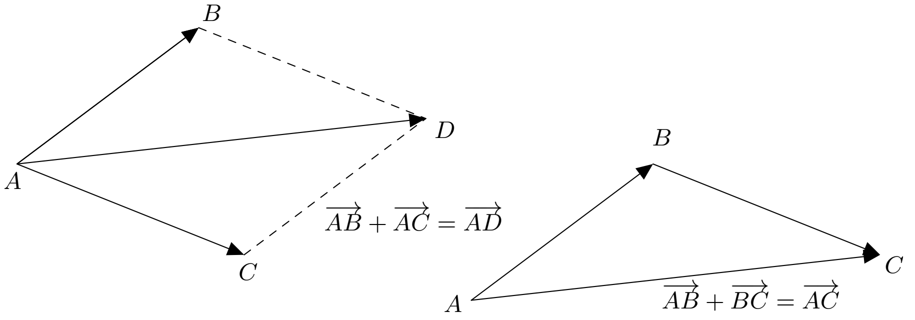

Figure 2

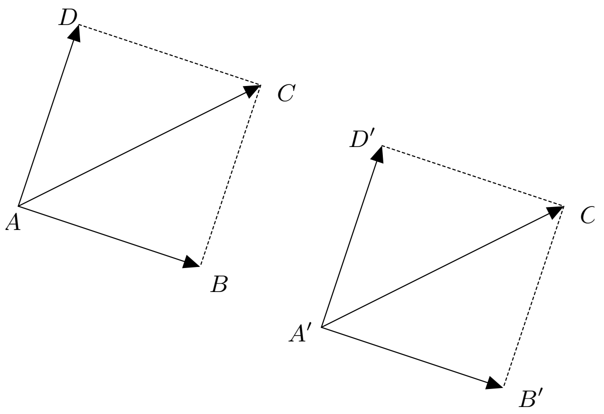

Figure 3

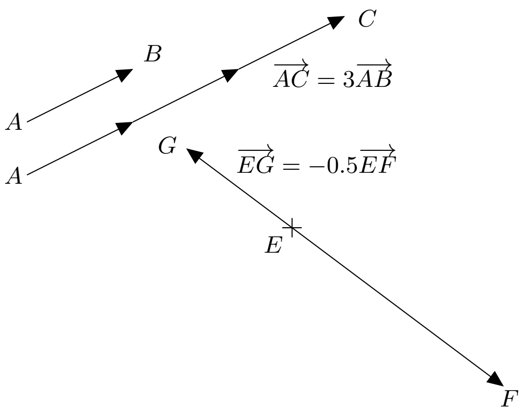

Figure 4

the two operations is the following two forms of the distributive law:

$$(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a}$$
$$\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$$

Both of them can be proved using directed segments as representatives of vectors to check these rules which belong to the required axioms of vector-space.

The most important thing is to see the difference between directed sections and vectors. They are on different levels of the building of mathematics. Helping later to understand the abstraction of vector space which is algebraically and not geometrically constructed. It is very useful to distinguish and abstract the notion of vector from geometry. This is impossible if we follow the classical false way to introduce the notion of the vector.

4. Summary and new results

Our aim was to show the similar mathematical connection between fractions and rational number respectively through directed segments and vectors. Behind both case the equivalence relation stands. We collected some examples which could help to deal earlier with this relation in school mathematics.

Our aim was to show how the notions of rational number and vector are taught traditionally. We found some typical misconceptions dealing with these notions in the school-mathematics that were illustrated by different textbooks from our countries. We think these textbook-examples are everywhere typical not only in Hungary and Slovakia. Finally we sketched some ideas about teaching these notions, first of all, *for teachers* and *text book authors* in the future. It would be necessary to plan experiments for students 10-13 years old to verify our thoughts. We hope, we will be able to report such experiments in the next future.

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